Critical k-Very Ampleness for Abelian Surfaces

CRITICAL k-VERY AMPLENESS FOR ABELIAN SURFACES

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Abstract. Let (S, L) be a $(1, d)$-polarized abelian surface with $c_1(L) = l$ and Picard rank 1. We define a function ϕ which takes each ample line bundle L into the least integer k such that L is k-very ample but not $(k+1)$-very ample. In this paper we use Bridgeland's stability conditions and Fourier-Mukai techniques to give a closed formula for $\phi(L^n)$ as a function of n. As a byproduct, we calculate the walls in the Bridgeland stability space for certain Chern characters.

1. Introduction

The notion of k-very ampleness was introduced in the 1980s initially to understand the idea of higher order embeddings. Weaker notions of k-spanned (see [BS91] and [BFS89]) and k-jet ampleness (see [BS93]) were also considered. The definitions given all relate to asking that, for a variety V and ample line bundle L on V, the natural map $\Gamma(L) \to \Gamma(L/\mathcal{I})$ surjects for certain classes of sheaf $\mathcal{I}$ of 0-subschemes of V. The notions differ in how fat points are treated. In this paper we only consider the strongest notion of k-very ampleness. Since k-very ampleness implies $(k-1)$-very ampleness it is natural to consider the critical value of k when a line bundle L is k-very ample but not $(k+1)$-very ample. We shall denote this by $\phi(L)$. In the case when the Neron-Severi group is generated by a single element (as we shall be assuming), we can view the function ϕ as a function of a positive integer. It is then natural to hope that $\phi(n+1)$ is related to $\phi(n)$ since $L^{n+1} = L^n \otimes L$. Unfortunately, k-very ampleness is not very well behaved with respect to tensoring.

One type of variety where most progress has been made is abelian varieties. There are a number of, by now, classical results on very ampleness of line bundles. For example, L^3 is always very ample for any ample line bundle L. A little more recently, Debarre, Hulek and Spandaw showed that a suitably generic $(1, d)$ polarization on a g-dimensional abelian variety is very ample for $d > 2^g$. For the case $g = 2$ and Picard rank 1, this was extended by Bauer and Szenberg ([BS97]) to compute $\phi(1)$ (see Proposition 4.1 below for the details).

There is a clear relation between k-very ampleness and so called "weak index theorem" conditions arising in Fourier-Mukai Theory for abelian varieties. These ideas have been extended by Popa and Pareschi ([PP03]) who introduce the notion of M-regularity and relate it to k-jet ampleness in [PP04].

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This paper is organized as follows. For the rest of the introduction we define the ϕ function and recall some facts already established in the literature. We also recall some facts about Fourier-Mukai transforms and deduce some easy results about ϕ. In section 2 we give a brief introduction to Bridgeland's stability conditions needed prove the main theorem in this paper. In section 3 we recall the notion of walls and show that, for the Chern character $(0, l, \chi)$ there are never any walls. We use this apply the general stability machinery to provide a useful technical lemma needed to prove our main Theorem. In the final section, we show how to use the technical lemma to bound ϕ from above and then prove that the bound is sharp by computing walls in the stability space associated to the Chern character $(1, nl, (n-1)^2d+d+1)$. We then induct on n to deduce the main theorem making use of our technical lemma again:

Theorem 4.3. *Let (S, L) be a $(1, d)$-polarized abelian surface with $NS(S) = \langle L\rangle$, then $\phi(L^n) = 2(n-1)d-2$.*

1.1. **k-Very Ample.** Let V be a complete algebraic variety of dimension g over an algebraic closed field $\mathbb{K}$, V a purely 0-dimensional subscheme of V with $|X| = d = \dim(H^0(\mathcal{O}_X))$ and L an invertible sheaf on V.

Definition 1.1. For each 0-scheme X on V we can consider the restriction map ρ_X to X for the space of sections of L, which fits into the exact sequence:

$$0 \to H^0(V, L \otimes \mathcal{I}_X) \to H^0(V, L) \overset{\rho_X}{\to} H^0(\mathcal{O}_X) \to H^1(V, L \otimes \mathcal{I}_X) \to H^1(V, L) \to 0$$

L is called k-very ample if ρ_X is surjective for all purely 0-dimensional subscheme X of length $|X| \leq k+1$.

Remark 1.2. The following follows easily from the definition

- L is 0-very ample if and only if L is generated by global section.
- L is 1-very ample if and only if it is very ample.
- If L is k-very ample then L is $(k-1)$-very ample.

Let $\mathrm{Amp}(S)$ be the ample cone of S. By the properties above there exists an integer k for all $L \in \mathrm{Amp}(S)$ such that L is k-very ample but not $(k+1)$-very ample.

Definition 1.3. Define a map

$$\phi : \mathrm{Amp}(S) \to \mathbb{Z}_{\geq -1}$$

which takes L into the least integer k such that L is k-very ample but not $(k+1)$-very ample. Define $\phi_L(n) := \phi(L^n)$, and $\phi(n)$ if L is understood.

There is no obvious reason why this should be a good function of n for any variety and, even for $\mathbb{P}^2$, it is hard to compute. Specific values for some varieties are, however, well known:

Example 1.4. Let (V, L) be a principally polarized abelian variety. Then $\phi_L(2) = 0$.

The following lemma, indirectly proved in [BS97] Proposition (3.2), gives the value of $\phi_L(1)$ and we will reprove it in §4 in the spirit of this paper:

Proposition 4.1. *If L is an ample line bundle of type $(1, d)$, $d \geq 1$ on an abelian surface X with Picard rank 1, then*

$$\phi_L(1) = \left\lfloor \frac{d-3}{2} \right\rfloor$$

Upper and lower bounds for ϕ are also known. It is clear that, if $H^1(L) = 0$ (as is the case, for example, for an ample line bundle on an abelian variety) then an upper bound for $\phi(n)$ can be given by $\chi(L^n) - 1$ since $\chi(L^n \otimes \mathcal{I}_X) = \chi(L^n) - |X|$. For a $(1, d)$ polarized abelian surface, this is $n^2 d - 1$. A non-trivial lower bound is much harder to come by but Reider's Theorem (for the most useful version, see [AB11, §2]) provides one, at least when certain divisors do not exist, as it says that if $c_1(L^n)^2 > (k+2)^2$ then L^n is k-very ample. If we apply this to the irreducible $(1, d)$ abelian surface case where such divisors do not exist, we see that $\phi(n) \geq \lceil \sqrt{d} n \rceil - 3$. But this is not even sharp for $d = 1$ and $n = 2$.

Reider's Theorem arose in the situation where $L \otimes \mathcal{I}_Z$ is used to construct vector bundles of rank 2. Key in his construction is the Bogomolov Inequality for semi-stable sheaves. We will also use this in various places and recall it here (see [HL10] for a proof).

Definition 1.5. A torsion-free sheaf E is μ-stable (μ-semistable) with respect to l if for each subsheaf F we have

$$\mu(F) < \mu(E) (\mu(F) \leq \mu(E))$$

where $\mu(E) = c_1(E).l^{g-1}/r(E)$.

Proposition 1.6. *Let V be a smooth projective variety of dimension n and l be an ample divisor on V. If E is a μ-semistable (with respect to l) torsion sheaf of rank r on V, then*

$$(r-1)c_1^2(E).l^{n-2} \leq 2rc_2^2(E).l^{n-2}.$$

For the case of $(1, d)$-polarization of abelian surface:

$$2r(E)\chi(E) \leq c_1^2(E).$$

We will also need to consider a finer stability for sheaves:

Definition 1.7. A torsion-free sheaf E is Gieseker stable (respectively Gieseker semistable) with respect to l if for each subsheaf F we have

$$P(F) < P(E) (P(F) \leq P(E))$$

where $P(E) = \dfrac{\chi(E \otimes L^n)}{r(E)}$, is the reduced Hilbert polynomial.

We let $\mathcal{M}_{\mathrm{ch}}^{\mathrm{GS}}$ denote the moduli space of Gieseker semistable sheaves on S with Chern character ch (or more generally, Simpson semistable sheaves when the rank is zero). These are known to be non-empty on an abelian surface whenever the Bogomolov inequality holds for ch.

1.2. **Fourier-Mukai transforms.** Let (V, Φ) be a smooth projective variety over $\mathbb{C}$ and $(\hat{S}, \hat{\Phi})$ be its dual. Consider flat projections $V \xleftarrow{\pi} V \times \hat{V} \xrightarrow{\hat{\pi}} \hat{V}$. Let $\mathcal{P} \in D(S \times \hat{V})$ where $D(V \times \hat{V})$ denotes the derived category of bounded complexes of coherent sheaves on $V \times \hat{V}$. The Fourier-Mukai transform Φ is the functor

$$\Phi : D(V) \to D(\hat{V}) \tag{1.1}$$

which takes $A^{\cdot}$ into $R\pi_*(L\hat{\pi}^* A^{\cdot} \overset{L}{\otimes} \mathcal{P})$ (See [Huy06]). Denote its cohomology by Φ^i. In fact, we shall only consider the classical Fourier-Mukai transform where P is the Poincaré bundle on and abelian surface $V = S$. Then Φ has a quasi-inverse given (up to shift) by the transform

$$\hat{\Phi} : D(\hat{V}) \to D(V) \tag{1.2}$$

with kernel $\hat{\mathcal{P}} \in D(\hat{S}) \times S$, where $\hat{\mathcal{P}} = s^*\mathcal{P}$ and $s : S \times \hat{S} \to \hat{S} \times S$ is $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Definition 1.8. An object E satisfies WIT_n if $\Phi^i(E) = 0$ for all $i \neq n$.

Definition 1.9. An object E satisfies IT_n if $H^i(E \otimes \mathcal{P}_{\hat{x}}) = 0$ for all $\hat{x} \in \hat{S}$ the dual of S and $i \neq n$. In which case, $\Phi^n(E)$ is a locally free sheaf.

Example 1.10. [Mum74] Any ample line bundle L on an abelian variety is IT_0. Any sheaf which is WIT_0 is automatically IT_0 by the semi-continuity theorem.

Proposition 1.11. *If (S, Φ) is an abelian variety and L is IT_0, then L is k-very ample if and only if $L \otimes \mathcal{I}_X$ is WIT_0 (and hence IT_0) for all 0-dimensional subschemes X of length $|X| \leq k+1$.*

Proof. " $\Rightarrow$ " Suppose that L is k-very ample and $L \otimes \mathcal{I}_X$ is not WIT_0 for some purely 0-dimensional subscheme X of length $|X| \leq k+1$, so there exists $\hat{x} \in \hat{S}$ such that $H^1(L\mathcal{P}_{\hat{x}}\mathcal{I}_X) \neq 0$. Let $x = \psi_L^{-1}(\hat{x})$ where $\psi_L : S \to \hat{S}$ takes x into $\tau_x L \otimes L^{-1}$, then $H^1(\tau_{-x}^*(L)\mathcal{I}_X) \neq 0$ and so $H^1(\tau_{-x}^*(L \otimes \mathcal{I}_{\tau_x X})) \neq 0$. Hence $H^1(L \otimes \mathcal{I}_{\tau_{-x}X}) \neq 0$ where $|\tau_{-x}X| \leq k+1$ and this contradicts the assumption.

" $\Leftarrow$ " Since $L \otimes \mathcal{I}_X$ is WIT_0 for all purely 0-dimensional subscheme X of length $|X| \leq k+1$, then $H^1(L \otimes \mathcal{I}_X) = 0$. Hence L is k-very ample by definition. □

Proposition 1.12. *Let (S, L) be an irreducible principally polarized abelian surface, then L^n is not $(2n-3)$-very ample.*

Proof. Let X be a 0-dimensional subscheme of D_L of length $2(n-1)$. Then we have a sequence

$$0 \to L^{n-1} \to L^n \otimes \mathcal{I}_X \to Q \to 0$$

Suppose Q is IT_0. The Chern character of Q is $\mathrm{ch}(Q) = (0, l, (2n-1)-|X|)$. Since $\hat{Q}$ the transform of Q has the Chern character $\mathrm{ch}(\hat{Q}) = ((2n-1)-|X|, -l, 0) = (1, -l, 0)$, but $\hat{Q}$ is locally-free which is impossible. So Q is not IT_0 and then $L^n \otimes \mathcal{I}_X$ is not IT_0. □

Such X we call collinear as $H^0(L \otimes \mathcal{I}_X) \neq 0$ so there exists $x \in S$ such that $X \subset \tau_x D_L$, a translation of the polarization divisor.

Corollary 1.13. *Let (S, L) be an irreducible principally polarized abelian surface, then $\phi(n) \leq 2n-4$ for $n \geq 2$.*

2. Bridgeland's Stabiliy Condition

Now we will give a brief review of Bridgeland's stability conditions (See [Bri08]. Define for any $s \in \mathbb{R}$ the following

$$F_s = \{E \in \mathrm{Coh}_S \,|E \text{ is torsion-free and } \mu_+(E) \leq 2ds\},$$

$$T_s = \{E \in \mathrm{Coh}_S \,|E \text{ is torsion or } \mu_-(E/\operatorname{tors}(E)) > 2ds\},$$

where $\mu_+(E)$ is the slope of the largest slope μ-destabilizing subsheaf of E and $\mu_-(E)$ is the slope of the lowest slope μ-destabilizing quotient of E. We set

$$\mathcal{A}_s = \{A \in D(S) | A^i = 0, i \notin \{0, -1\}, H^{-1}(A) \in F_s, H^0(A) \in T_s\}.$$

A group homomorphism $Z_{s,t}$ takes the Chern character $\mathrm{ch}(A)$ into

$$\begin{aligned} Z_{s,t}(A) &= \left\langle e^{(s+ti)l}, ch(A) \right\rangle \\ &= -\chi - +2dcs + dr(t^2 - s^2) + 2tdi(c - rs). \end{aligned}$$

For each $A \in \mathcal{A}_s$ the slope $\mu_{s,t}(A)$ of A is given by:

$$\mu_{s,t}(A) = -\frac{\mathrm{Re}(Z_{s,t}(A))}{\mathrm{Im}(Z_{s,t}(A))} \tag{2.1}$$

$$= \frac{\chi - 2dcs - dr(t^2 - s^2)}{2td(c - rs)}. \tag{2.2}$$

Definition 2.1. We say that $E \in \mathcal{A}_s$ is σ_t-stable (respectively, σ_t-semistable) if for all proper injections $F \to E$ in $\mathcal{A}_s$ we have $\mu_{s,t}(F) < \mu_{s,t}(E)$ ($\mu_{s,t}(F) \leq \mu_{s,t}(E)$, respectively). It is well known that these give sensible stability conditions on any smooth surface.

Then the slope of $E = L^n \otimes \mathcal{I}_X$ where $ch(E) = (1, nl, n^2 d - |X|)$ is

$$\mu_{s,t}(E) = \frac{n^2 d - |X| - 2dns - d(t^2 - s^2)}{2td(n - s)}.$$

Note that $n > s$ as $E \in T_s$.

Remark 2.2. Now suppose $F \in \mathcal{A}_s$ with $\operatorname{ch}(F) = (r, cl, \chi)$ destabilizes $L^n \otimes \mathcal{I}_X$. The we have a short exact sequence $F \to E \to Q$ in $\mathcal{A}_s$. Taking cohomology we see that $H^{-1}(F) = 0$. Then $F \in T_s$ and so $c > rs$. Notice also that $H^{-1}(Q) \in F_s$ is torsion-free and since $0 \to H^{-1}(Q) \to F \to E$ is exact, F is also torsion-free.

We also have

$$\mu_{s,t}(F) - \mu_{s,t}(E) \geq 0.$$

Therefore

$$\frac{\chi - 2dcs + dr(s^2 - t^2)}{2td(c - rs)} - \frac{n^2d - |X| - 2dns - d(t^2 - s^2)}{2td(n - s)} \geq 0. \tag{2.3}$$

Define $f(F, E)$ to be the numerator of (2.3), then

$$\begin{aligned} f(F,E) &= (\chi - 2dcs + dr(s^2 - t^2))(n - s) \\ &\qquad - (n^2d - |X| - 2dns - d(t^2 - s^2))(c - rs) \\ &= (n - s)\chi - c\big(2ds - n^2d + |X| + 2dns + d(s^2 + t^2)\big) \\ &\qquad + r\big(n^2ds + |X|s - nd(s^2 + t^2)\big). \end{aligned}$$

We shall be most interested in the case when $s = 0$. Then the destabilizing condition becomes

$$f(F, E) = n\chi - c(n^2d - |X| - dt^2) - dnrt^2 \geq 0. \tag{2.4}$$

Therefore

$$n\chi - cn^2d + c|X| \geq (nr - c)dt^2,$$

and $c \leq nr$ because $\mu(H^{-1}(Q)) \leq 0$ and $\mu(F) \leq \mu(F/H^{-1}(Q)) \leq \mu(E)$. Hence a necessary condition for the existence of such a destabilizing object is

$$n\chi - cn^2d + c|X| > 0. \tag{2.5}$$

Recall from [Bri08, Prop 14.2] that in the "large volume limit" as $t \to \infty$, the σ_t-semistable objects E with $\mu(E) > 0$ are exactly the Gieseker semistable sheaves (when $s = 0$). The case when $\mu(E) < 0$ is similar:

Proposition 2.3. *For all $t \gg 0$. $F \in \mathcal{A}_0$ with $\mu(F) < 0$ is σ_t-semistable and $\mu(F) < 0$ if and only if $H^0(F)$ is supported in dimension 0 and $H^{-1}(F)$ is Gieseker semistable vector bundle.*

Proof. Proof follows in same way as that of [Bri08, Prop 14.2] by observing that if E is Bridgeland stable for all $t \gg 0$ then $H^0(E)$ must be supported in dim 0, otherwise $\mu_{0,t}(H^0(E))$ is finite and $H^0(E)$ destabilizes E for $t >> 0$. Moreover $H^{-1}(E)[1]$ is locally free since

$$0 \to \mathcal{O}_Z \to H^{-1}(E)[1] \to H^{-1}(E)^{**}[1] \to 0$$

is short exact sequence in $\mathcal{A}_0$ and then $\mathcal{O}_Z \to E$ would destabilize E. The fact that $H^{-1}(E)$ is Gieseker semistable follows in the same way as [Bri08]. $\square$

Remark 2.4. An alternative approach can be seen using an observation of Yanagida and Yoshioka who show that the Bridgeland stability is preserved under $[1] \circ \Delta$, where $\Delta(E) = \mathbf{R}\mathcal{H}om(E, \mathcal{O}_S)$ (see [YY12, Prop 2.9]) at least when $c_1 \cdot \ell \neq 0$. So if F is σ_t-semistable then $F^\vee$ is σ_t-semistable and $\mu(F^\vee) > 0$. Then Prop. 14.2 in [Bri08] implies that $F^\vee$ is G-semistable sheaf. Therefore $F^{\vee\vee} \cong F$ takes required form. In particular, observe that $H^0(F) \neq 0$ exactly when $F^\vee$ is not locally-free.

Remark 2.5. Huybrechts ([Huy08]) showed that $\Phi[1]$ preserves $\mathcal{A}_0$ and it can also be shown (see for example, [MM12]) that $E \in \mathcal{A}_0$ is σ_t-stable if and only if $\Phi(E)$ is $\sigma_{1/t}$-stable (and similarly for semistable).

3. Walls and Moduli Spaces

Definition 3.1. We let $\mathcal{M}^{\mathrm{BS},t}_{\mathrm{ch}}$ denote the moduli space of σ_t-semistable objects in $\mathcal{A}_0$. It is now well known that these exist (when non-empty) (see, for example [MM12]).

For example, in the large volume limit as $t \to \infty$, $\mathcal{M}^{\mathrm{BS},t}_{\mathrm{ch}} = \mathcal{M}^{\mathrm{GS}}_{\mathrm{ch}}$ when $c_1(\mathrm{ch})\cdot l > 0$. Equality here means that the points represent exactly the same objects in $\mathrm{Coh}(S) \cap \mathcal{A}_0$ up to isomorphism.

It may happen for some value of t that the two moduli spaces are not equal. In fact, there will be a strictly decreasing sequence $t_0, t_1, \ldots$ of values of t on either side of which $\mathcal{M}^{\mathrm{BS},t}_{\mathrm{ch}}$ differ. We call these walls. (sometimes they are called mini-walls when we fix s). Our aim will be to identify these walls when $\mathrm{ch} = (0, l, \chi)$ and $\mathrm{ch} = (1, nl, n^2d - |x|)$. In the first case we show there are no walls. The following generalizes [Mac12, Prop 4.2]:

Lemma 3.2. *For $s = 0$ there are no walls for $\mathrm{ch} = (0, l, \chi)$ for any $\chi \in \mathbb{Z}$.*

Proof. Let $F \in T_0$ with $\mathrm{ch}(E) = (r, cl, k)$ and $c \geq 0$ destabilize $E \in \mathcal{M}^{\mathrm{GS}}_{(0,l,\chi)}$. Then we have the following exact sequence in $\mathrm{Coh}(X)$

$$0 \to F' \to F \to E \to Q \to 0, \tag{3.1}$$

where $F' \in F_0$. Factoring $F \to E$ via $K \hookrightarrow E$, we see that $K \neq 0$ implies that $c_1(K) = l$ as E is pure. So $c_1(F') = (c-1)l$. Then Q is supported in dimension 0. Let $\chi(Q) = a \geq 0$ then $\mathrm{ch}(F') = \big(r, (c-1)l, k - \chi + a\big)$. Since $F' \in F_0$, then $c \leq 0$. Therefore $c = 1$. From (2.2), the stability condition is given by $k > \chi$. On the other hand, F' is G-stable by Proposition 2.3 , then $(k - \chi + a)r \leq 0$ so $k < \chi$. Hence there is no walls. □

Remark 3.3. In particular, by Remark 2.5, $\mathrm{ch} = (\chi, -l, 0)$ and $\mathrm{ch} = (\chi, l, 0)$ also have no walls. Hence, for all $t > 0$

$$\mathcal{M}^{BS,t}_{(\chi,l,0)} = \mathcal{M}^{GS}_{(\chi,l,0)}.$$

Definition 3.4. We shall say that the moduli space $\mathcal{M}^{BS,t}_{(r,cl,\chi)}$ of Bridgeland stable sheaves of Chern character (r, cl, χ) satisfies IT_0 (respectively WIT_0) if and only if for each E representing an object of $\mathcal{M}^{BS}_{(r,cl,\chi)}$, E satisfies IT_0 (respectively WIT_0).

For example $\mathcal{M}^{BS,t}_{(1,nl,n^2d-k)}$ is IT_0 for all t if and only if L^n is $(k-1)$-very ample and so $\phi_L(n) \geq k-1$.

The following technical result will be useful in the next section:

Lemma 3.5. *$\mathcal{M}^{\mathrm{GS}}_{(0,l,\chi)}$ is IT_0 if and only if $\chi \geq d+1$.*

Proof. We use Proposition 2.3, Remarks 2.4 and 3.3, and Lemma 3.2 to give isomorphisms

$$\mathcal{M}^{\mathrm{GS}}_{(0,l,\chi)} \overset{\Phi[1]}{\cong} \mathcal{M}^{\mathrm{BS},t}_{(\chi,-l,0)} \overset{[1]\Delta}{\cong} \mathcal{M}^{\mathrm{BS},t}_{(\chi,l,0)} = \mathcal{M}^{\mathrm{GS}}_{(\chi,l,0)}$$

for all $t > 0$. Then $[E] \in \mathcal{M}^{\mathrm{GS}}_{(0,l,\chi)}$ is IT_0 if and only of $[\Phi(E)[1]] \in \mathcal{M}^{\mathrm{BS},t}_{(\chi,-l,0)} \cap \mathcal{M}^{\mathrm{GS}}_{(\chi,-l,0)}[1]$ which holds if and only if $\Delta\Phi(E) \in \mathcal{M}^{\mathrm{GS}}_{(\chi,l,0)}$ is locally-free. But, since all representative sheaves of $\mathcal{M}^{\mathrm{GS}}_{(\chi,l,0)}$ must be μ-stable, we see that there are non-locally-free sheaves in $\mathcal{M}^{\mathrm{GS}}_{(\chi,l,0)}$ if and only if $\mathcal{M}^{\mathrm{GS}}_{(\chi,l,1)} \neq \emptyset$. This happens exactly when the Bogomolov inequality fails for the Chern character $(\chi, l, 1)$, in other words when $\chi \leq d$ as required. $\square$

4. $(1, d)$-Polarization Line Bundles

Let L be $(1, d)$-polarization line bundle on an abelian surface S with $c_1(L) = l$ and $l^2 = 2d$. In this section we will prove some lemmas that help us to find the value of $\phi_L(n)$.

Proposition 4.1. *If L is an ample line bundle of type $(1, d)$, $d \geq 1$ on an abelian surface X, then*

$$\phi_L(1) = \left\lfloor \frac{d-3}{2} \right\rfloor$$

Proof. The Chern character of $E = L \otimes \mathcal{I}_X$ is $\mathrm{ch}(L \otimes \mathcal{I}_X) = (1, l, d - |X|)$. Then $\mathrm{ch}(\Phi(E)) = (d - |x|, -l, 1)$. Such objects are all locally-free sheaves exactly when there are no stable sheaves with Chern character $(d-|x|, -l, 2)$. These exist exactly when the Bogomolov inequality holds for such a Chern character (see Definition 3.1). This gives us the criterion $2(d - |X|) \leq d$, so $|X| \leq \left\lceil \frac{d}{2} \right\rceil = \left\lfloor \frac{d-1}{2} \right\rfloor$. Hence $\mathcal{M}^{\mathrm{GS}}_{(1,l,d-|X|)}$ is IT_0 if and only if $|X| \leq \left\lfloor \frac{d-1}{2} \right\rfloor$. Then $\phi_L(1) = \left\lfloor \frac{d-3}{2} \right\rfloor$. $\square$

Proposition 4.2. *Let (S, L) be an irreducible $(1, d)$-polarized abelian surface, then $\phi_L(n) \leq 2(n-1)d - 2$ for $n > 1$.*

Proof. By Lemma 3.5, there is Q with Chern character $\mathrm{ch}(Q) = (0, l, d)$ which is not IT_0. Since $\chi(L^{-n+1} \otimes Q) = d(3 - 2n) < 0$ for $n > 1$ so $\mathrm{Ext}^1(Q, L^{n-1}) \neq 0$. Pick a non trivial extension

$$0 \to L^{n-1} \to E \to Q \to 0$$

and suppose $T \hookrightarrow E$ is its torsion subsheaf. Then we have the following diagram:

$$\begin{tikzcd}
 & & F \arrow[r] & Q/T & \\
0 \arrow[r] & L^{n-1} \arrow[r] \arrow[ur] & E \arrow[r] \arrow[u] & Q \arrow[r] \arrow[u] & 0 \\
 & & T \arrow[u] \arrow[r, equal] & T \arrow[u] &
\end{tikzcd}$$

Then Q/T must be supported in dimension zero. But then $\mathrm{Ext}^1(Q/T, L^{n-1}) = 0$ and so $L^{n-1} \to F \to Q/T$ must split, which is impossible. Hence $T = 0$. Then $E \cong L^n \otimes \mathcal{I}_X$ for some X of length $|X| = 2d(n-1)$ and E is not IT_0. $\square$

The following theorem proves that the upper pound of $\phi_L(n)$ in Proposition 4.2 is sharp.

Theorem 4.3. *Let (S, L) be a $(1, d)$ polarized abelian surface with* $\mathrm{NS}(S) = \langle L \rangle$, *then $\phi(L^n) = 2d(n-1) - 2$.*

Proof. By Proposition 4.2 we need to show that $\phi_L(n) \geq 2(n-1)d - 2$, and we do this by showing that $\mathcal{M}^{\mathrm{BS},t}_{(1,nl,n^2d-k)}$ is IT_0 for all t and $k = 2d(n-1) - 1$. Suppose that $E \cong L^n \otimes \mathcal{I}_X$ where $|X| = 2d(n-1) - 1$ is not IT_0 and $\Phi(E)$ is σ_t-stable for all $t \gg 0$. Then $\hat{E}$ is a two-step complex such that $H^{-1}(\hat{E})$ is G-stable and $H^0(\hat{E})$ is in the form $\mathcal{O}_Z$, by Proposition 2.3. The Chern character $\mathrm{ch}(H^{-1}(\hat{E})) = ((n-1)^2 d + d + 1, n, 1 + |Z|)$. By Bogomolov

$$\big((n-1)^2 d + d + 1\big)(1 + |Z|) \leq n^2 d. \tag{4.1}$$

Therefore $|Z| \leq \dfrac{2d(n-1) - 1}{dn^2 - 2d(n-1) + 1}$. But $d(n-2)^2 + 2 > 0$ and so

$$dn^2 - 2d(n-1) + 1 > 2d(n-1) - 1.$$

Hence, $|Z| < 1$. Therefore $H^0(\hat{E}) = 0$ and so E is IT_0. If E is σ_t-stable for all t, then it follows that $\Phi(E)$ is σ_t-stable for all t (and so also for $t \gg 0$). This happens when there are no walls. Unfortunately, there are walls in general. To finish the proof we will identify all the walls and show that all σ_t-semistable objects are IT_0 directly.

Lemma 4.4. *If $e \in \mathcal{A}_0$ destabilizes $L^n \otimes \mathcal{I}_X$ with $|X| = 2d(n-1) - 1$, then e is a rank 1 torsion-free sheaf.*

Proof. By Remark 2.2, $H^{-1}(e) = 0$ and $E := H^0(e) \cong e$ is torsion-free. Suppose $\mathrm{ch}(E) = (r, g'l, \chi)$ and let $q = L^n \otimes \mathcal{I}_X/E$ in $\mathcal{A}_0$. Then we have a long exact sequence in $\mathrm{Coh}(S)$:

$$0 \to H^{-1}(q) \to E \to L^n \otimes \mathcal{I}_X \to H^0(q) \to 0 \tag{4.2}$$

Since $H^{-1}(q) \in \mathcal{F}_0$ and $E \in \mathcal{A}_0$, then $\mu(E) \geq 0 \geq \mu(H^{-1}(Q))$. Then there is an integer $g > 0$ such that $c_1(E) = (nr-g)\ell$. Then $c_1(H^{-1}(Q)) = (nr-g-n+m)\ell \leq 0$ where $m \geq 0$. Therefore $0 < nr - g \leq n - m \leq n$. Hence, $c_1(E)$ can be written as $c_1(E) = (n-c)\ell$ for some positive integer $c < n$. Since any destabilizer of a sheaf can be taken as a G-stable, the Bogomolov inequality allow us to write

$$\chi(E) = \frac{(n-c)^2 d}{r} - k,$$

for some rational number $k \geq 0$. Since E is a destabilizer of $L^n \otimes \mathcal{I}_X$, we have $f(E, L^n \otimes \mathcal{I}_X) > 0$. Therefore from a condition (2.5), we get

$$\big((n-c)^2 d - kr\big)n - (n^2 d - 2dn + 2d + 1)(n-c)r > 0, \tag{4.3}$$

Rearrange (4.3), we obtain

$$(n-c)\big(-(n-1)^2 dr - dr - r + dn^2 - cdn\big) > krn > 0 \tag{4.4}$$

As $n - c > 0$, then we get walls if $-(n-1)^2 dr - dr - r + dn^2 - cdn > 0$ so

$$\frac{1}{r} > \left(1 - \frac{1}{n}\right)^2 + \frac{1}{n^2} + \frac{1}{dn^2} \geq \frac{1}{2},$$

for all n, since $d > 0$. Hence $r = 1$. □

Remark 4.5. The previous lemma proved that the Chern character of any destabilizer of $L^n \otimes \mathcal{I}_X$ is given by $\mathrm{ch}(E) = (1, n-c, (n-c)^2 d - k)$ which means that E is in the form $L^{n-c} \otimes \mathcal{P}_{\hat{x}} \otimes \mathcal{I}_Y$, for some $\hat{x} \in \hat{S}$ and $|Y| = k$.

Lemma 4.6. *If $L^{n-m} \otimes \mathcal{P}_{\hat{x}} \otimes \mathcal{I}_Y$ destabilizes $L^n \otimes \mathcal{I}_X$ where $|X| = 2d(n-1) - 1$, then $m = 1$.*

Proof. We assume, without loss of generality, that $\hat{x} = 0$. Suppose that $F = L^{n-m} \otimes \mathcal{I}_Y$ with $|Y| = k$ is destabilizing $E = L^n \otimes \mathcal{I}_X$, then $\mu(F) - \mu(E) > 0$. Therefore from a condition (4.4), we get

$$(n-m)(-(n-1)^2 dr - dr - r + dn^2 - dnm) > krn.$$

Then we get walls if and only if

$$\left(1 - \frac{m}{n}\right)(2dn - nmdr - 2d) > k \geq 0. \tag{4.5}$$

Since $1 - \dfrac{m}{n}$ is positive, this happens if and only if $2dn - nmdr - 2d > 0$ and then $2 \geq 2 - \dfrac{d-1}{nd} > m > 0$. Hence, $m = 1$. □

Lemma 4.7. *If* $L^{n-1} \otimes \mathcal{P}_{\hat{x}} \otimes \mathcal{I}_Y$ *destabilizes* $L^n \otimes \mathcal{I}_X$ *for some* X *where* $|X| = 2d(n-1)-1$*, then* $|Y| < d(n-2) \leq 2d(n-2)-1$.

Proof. Without loss of generality we assume $\hat{x} = 0$. Take F, E as Lemma 4.6, then from 4.5 we get:

$$\left(1 - \frac{1}{n}\right)(dn - 2d) > |Y| \geq 0 \tag{4.6}$$

Since $0 < 1 - \dfrac{1}{n} < 1$, then $dn - 2d > |Y|$. □

To complete proof of Theorem 4.3, we now induct on $n \geq 2$. If $n = 2$, then $d(n-2) = 0$ and so there are no walls which establishes the result for $n = 2$

Suppose that the statement is true for $n - 1 \geq 2$. i.e. $L^{n-1} \otimes \mathcal{I}_X$ is IT_0 for all X with $|X| = 2d(n-1) - 1$ To prove that $L^n \otimes \mathcal{I}_X$ is IT_0 for all X with $|X| = 2d(n-1) - 1$, we know that the only possible walls are given by $L^{n-1} \otimes \mathcal{I}_Y$ where $|Y| < 2d\big((n-1)-1\big) - 1$. Then there is a short exact sequence

$$0 \to L^{n-1} \otimes \mathcal{I}_Y \to L^n \otimes \mathcal{I}_X \to Q \to 0$$

By induction, $L^{n-1} \otimes \mathcal{I}_Y$ is IT_0 and by Lemma 3.5, Q is IT_0 as well, since

$$\chi(Q) = -(n-1)^2 d + |Y| + n^2 d - 2d(n-1) + 1 = 1 + d + |Y| \geq d + 1.$$

Hence $L^n \otimes \mathcal{I}_X$ is IT_0 for all X with $|X| = 2d(n-1) - 1$. □

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